A NOTE ON BAYES ESTIMATORS AND ROBUSTNESS OF LOGNORMAL PARAMETERS

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1. The lognormal distribution arises in various different contexts such as Physics (distribution of small particles), Economics (income distribution), Biology (growth of organisms), etc. A comprehensive treatment of lognormal distribution has been given by Aitchison and Brown [1]. Epstein [5], Brownlee [3], Delaporte [4], Moroney [9] describe applications of lognormal distribution to physical and industrial processes, textile research and quality control. In the context of life testing problems, the lognormal distribution answers a criticism sometimes raised against the use of normal distribution (ranging from $-\infty$ to $+\infty$) as a model for failure time distribution which must range from 0 to ∞.

Consider the lognormal probability density function (pdf)

$$f(x \mid x_0, \mu, \sigma)=\frac{1}{\sqrt{2\pi}\,\sigma\,(x-x_0)}$$

$$\exp\left[\,1-\frac{1}{2\sigma^2}\,(\log\,(x-x_0)-\mu)^2\,\right]\; x>x_0 \qquad \ldots(1)$$

Hill [7] has shown that there exists paths along which the likelihood function of a sample

$$(x_1, x_2, \ldots\, x_n)\rightarrow\infty \text{ as } (x_0, \mu, \sigma)\rightarrow\{x_{(1)}, -\infty, \infty\}$$

where $x_{(1)}$ is the smallest of the x_i and hence [in a meaningful sense] these are the maximum likelihood estimators. Giesbrecht and Kempthorne [6] obtained the maximum likelihood estimators when the data from the pdf (1) are grouped. Sinha [10] suggested an easy way to compute moment estimators.

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Assuming x_0 known, we will study the robustness of Bayes estimators of $\theta \equiv (\mu, \sigma)$ and the corresponding posteriors when one has little or vague prior information about the parameters. In such situations Jeffreys [8] proposed the prior $p(\theta) \propto | I(\theta) |^{\frac{1}{2}}$ where $I(\theta)$ is Fisher's information matrix. Use of Jeffreys prior has resulted in a number of interesting and well known estimators.

Let $\quad x - x_0 = y.$

We have
$$f(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\, \sigma y} \exp\left\{-\frac{1}{2\sigma^2}[(\log y) - \mu]^2\right\}, \; y > 0.$$

It is easy to show that Jeffreys prior $p(\mu, \sigma) \propto \frac{1}{\sigma^2}$.

Consider the class of 'improper' or 'quasi' priors $p(\mu, \sigma) \propto \frac{1}{\sigma^c}$, $\sigma, c > 0$. Given the data $\underline{y} = (y_1, y_2, \ldots y_n)$, the likelihood function $L(\mu, \sigma \mid \underline{y})$, and the prior $p(\mu, \sigma)$ and making use of Bayes theorem [2], we have the posterior distribution

$$\Pi(\mu, \sigma \mid \underline{y}) = K\, p(\mu, \sigma)\, L(\mu, \sigma \mid \underline{y})$$

where K is a normalizing constant.

2. The joint posterior of (μ, σ) is given by

$$\Pi(\mu, \sigma \mid \underline{y}) = \frac{K}{\sigma^{n+c}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\log y_i - \mu)^2\right\}$$

$$= \frac{K}{\sigma^{n+c}} \exp\left(-\frac{\lambda}{2\sigma^2}\right) \exp\left\{-\frac{n}{2\sigma^2}\left(\mu - \frac{\sum_{i=1}^{n} \log y_i}{n}\right)^2\right\}$$

where
$$\lambda = \sum_{i=1}^{n} (\log y_i)^2 - \frac{\left(\sum_{i=1}^{n} \log y_i\right)^2}{n}.$$

The marginal posterior of σ is given by

$$\Pi(\sigma \mid \underline{y}) = \frac{K}{\sigma^{n+c}} \exp\left(\frac{\lambda}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left\{-\frac{n}{2\sigma^2}\left(\mu - \frac{\sum_{i=1}^{n} \log y_i}{n}\right)^2\right\} du$$

$$= \frac{\lambda^{\frac{n+c-2}{2}} \exp\left(-\frac{\lambda}{2\sigma^2}\right)}{2^{\frac{n+c-4}{2}}\, \Gamma\frac{n+c-2}{2}\, \sigma^{n+c-1}}, \quad \sigma > 0$$

and similarly

$$\Pi(\mu \mid \underline{y})$$

$$= \sqrt{\frac{n}{\lambda}}\, \frac{1}{B\left(\frac{1}{2}, \frac{n+c-2}{2}\right)\left\{1 + \frac{n}{\lambda}\left(\mu - \frac{\sum_{i=1}^{n} \log y_i}{n}\right)^2\right\}^{\frac{n+c-1}{2}}} \quad -\infty < \mu < \infty$$

Bayes estimator σ^*

$$= E(\sigma \mid \underline{y}) = \left(\frac{\lambda}{2}\right)^{\frac{n+c-2}{2}} \frac{1}{\Gamma\left(\frac{n+c-2}{2}\right)} \int_{0}^{\infty} \frac{\exp\left(-\frac{\lambda}{2\sigma^2}\right) d\sigma^2}{(\sigma^2)^{\frac{n+c-3}{2}+1}}$$

$$= \frac{\lambda}{2}\, \frac{\Gamma\left(\frac{n+c-3}{2}\right)}{\Gamma\left(\frac{n+c-2}{2}\right)}, \text{ a function of } c. \qquad \ldots(2)$$

while

$$\mu^* = \sqrt{\frac{n}{\lambda}}\, \frac{1}{B\left(\frac{1}{2}, \frac{n+c-2}{1}\right)} \int_{-\infty}^{\infty} \frac{\mu\, d\mu}{\left\{1 + \frac{n}{\lambda}\left(\mu - \frac{\sum_{i=1}^{n} \log Y_i}{n}\right)^2\right\}^{\frac{n+c-1}{2}}}$$

$$= \frac{\sum_{i=1}^{n} \log Y_i}{n} \qquad \ldots(3)$$

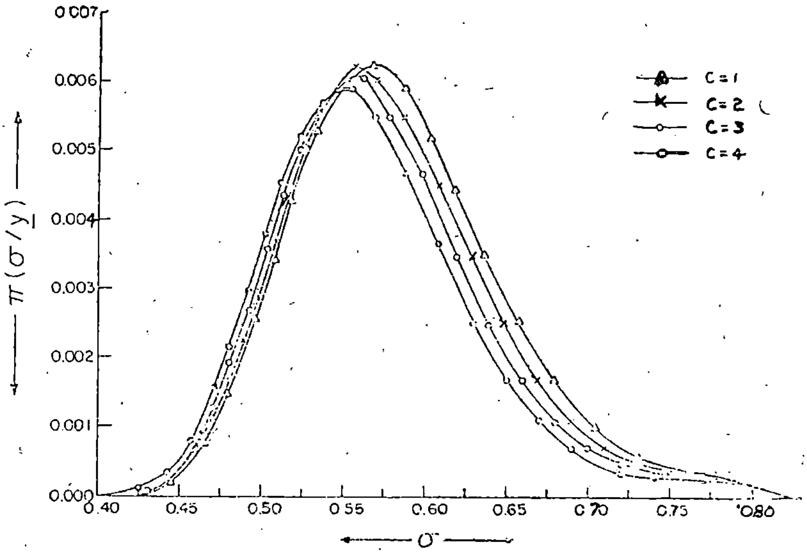

Fig. 1. Posterior Distribution of σ

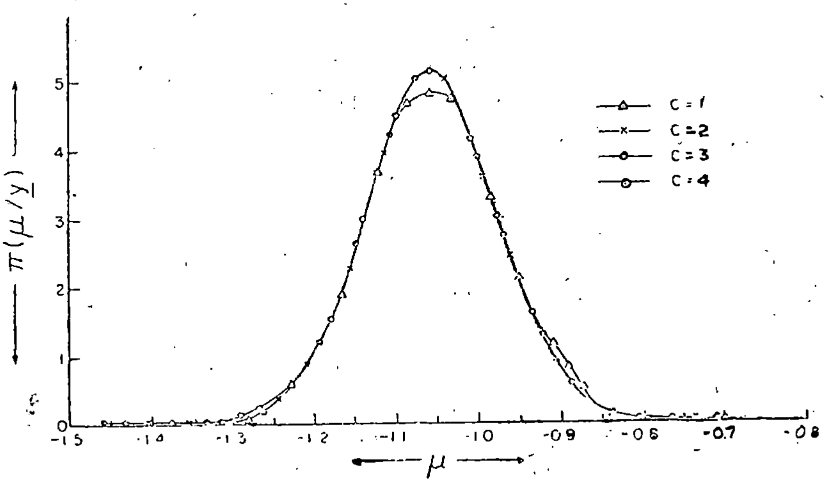

Fig. 2. Posterior Distribution of μ $g(\mu, \sigma) \propto \frac{1}{\sigma^{c}}$

an expression independent of c, which shows that under the joint prior $p(\mu, \sigma) \propto \frac{1}{\sigma^c}$, Bayes estimator of μ is uniformly robust for changes in c but that of σ is less so.

A random sample of 50 was generated from the pdf(1) with $x_0=6.0$, $\mu=-1.0$, $\sigma=0.5$.

Using (2) and (3) we obtain Bayes estimators σ^*, μ^*.

c	σ^*
1	0.577
2	0.571
3	0.565
4	0.560

$$\frac{\text{Min}(\sigma^*)}{\text{Max}(\sigma^*)}=0.97$$

True $\sigma=0.5$. $\mu^*=-1.054$. True $\mu=-1.0$.

The posteriors $\Pi(\sigma \mid \underline{y})$ and $\Pi(\mu \mid \underline{y})$ are plotted in figures 1 and 2. The estimates σ^* and μ^* justify the patterns of the corresponding posteriors.

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